

太原市 2021 年高三年级模拟考试（二）

数学试题（理）参考答案及评分标准

一. 选择题: A D C C B B A D C B C C

二. 填空题: 13. 60° 14. $\frac{7}{9}$ 15. $2\ln 2$ 16. $[\frac{9}{4}, 3)$

三. 解答题:

$$17. (I) \text{证明: } \because \frac{a_{n+1}}{n(a_{n+2}-a_{n+1})} = \frac{a_n}{(n+1)(a_{n+1}-a_n)} + \frac{1}{2n(n+1)}, \therefore \frac{(n+1)a_{n+1}}{a_{n+2}-a_{n+1}} = \frac{na_n}{a_{n+1}-a_n} + \frac{1}{2},$$

$$\therefore b_{n+1} = b_n + \frac{1}{2}, \therefore b_{n+1} - b_n = \frac{1}{2} \text{ 是一个与 } n \text{ 无关的常数,} \dots\dots\dots 4 \text{ 分}$$

$$\therefore \{b_n\} \text{ 是以首项 } b_1 = \frac{1}{2} \text{、公差 } d = \frac{1}{2} \text{ 的等差数列;} \dots\dots\dots 6 \text{ 分}$$

$$(II) \text{ 由 } (I) \text{ 得 } b_n = \frac{n}{2} (n \in N^*), \therefore \frac{na_n}{a_{n+1}-a_n} = \frac{n}{2}, \therefore a_{n+1} = 3a_n,$$

$$\because a_1 = 1, \therefore a_n = 3^{n-1} (n \in N^*), \therefore c_n = \frac{b_n}{a_n} = \frac{n}{2 \times 3^{n-1}} (n \in N^*), \dots\dots\dots 8 \text{ 分}$$

$$\therefore S_n = \frac{1}{2} \times (1 + \frac{2}{3} + \frac{3}{3^2} + \dots + \frac{n}{3^{n-1}}), \quad ①$$

$$① \times \frac{1}{3} \text{ 得 } \frac{1}{3} S_n = \frac{1}{2} \times (\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots + \frac{n-1}{3^{n-1}} + \frac{n}{3^n}), \quad ②$$

$$① - ② \text{ 得 } \frac{2}{3} S_n = \frac{1}{2} \times (1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^{n-1}} - \frac{n}{3^n}),$$

$$\therefore S_n = \frac{1}{8} \times (9 - \frac{2n+3}{3^{n-1}}). \dots\dots\dots 12 \text{ 分}$$

18 解: (I) 由频率分布直方图得该样本中垃圾量为 $[4,6)$, $[6,8)$, $[8,10)$, $[10,12)$, $[12,14)$, $[14,16)$, $[16,18]$ 的频率分别为 0.08, 0.1, 0.2, 0.24, 0.18, 0.12, 0.08,

$\bar{x} = 5 \times 0.08 + 7 \times 0.10 + 9 \times 0.20 + 11 \times 0.24 + 13 \times 0.18 + 15 \times 0.12 + 17 \times 0.08 = 11.04 \approx 11$,
所以当天这 50 个社区垃圾量的平均值为 11 吨; $\dots\dots\dots 4 \text{ 分}$

(II) 由 (I) 知 $\mu = 11$, $\therefore \sigma^2 = 9$, $\therefore \sigma = 3$,

$$\therefore P(X > 14) = P(X > \mu + \sigma) = \frac{1 - 0.6827}{2} = 0.15865,$$

所以这 200 个社区中“超标”社区的个数为 $200 \times 0.15865 \approx 32$; $\dots\dots\dots 7 \text{ 分}$

(III) 由 (I) 得样本中当天垃圾量为 $[14,16)$ 的社区有 $50 \times 0.12 = 6$ 个, 垃圾量为 $[16,18]$ 的社区有 $50 \times 0.08 = 4$ 个, 所以 Y 的可能取值为 0, 1, 2, 3, $\dots\dots\dots 8 \text{ 分}$

$$P(Y=0) = \frac{C_6^3}{C_{10}^3} = \frac{1}{6}, \quad P(Y=1) = \frac{C_6^2 C_4^1}{C_{10}^3} = \frac{1}{2}, \quad P(Y=2) = \frac{C_6^1 C_4^2}{C_{10}^3} = \frac{3}{10}, \quad P(Y=3) = \frac{C_4^3}{C_{10}^3} = \frac{1}{30},$$

$\therefore Y$ 的分布列为

Y	0	1	2	3
P	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

$$\therefore EY = 0 \times \frac{1}{6} + 1 \times \frac{1}{2} + 2 \times \frac{3}{10} + 3 \times \frac{1}{30} = \frac{6}{5}. \quad \dots\dots\dots 12 \text{ 分}$$

19. (I) 证明: 设点 G, H 分别是 CD, CB 的中点, 连结 EG, FH, GH ,

则 $GH \parallel DB$, 且 $DB = 2GH$, $\therefore EF \parallel DB$, 且 $DB = 2EF$, $\therefore EF \parallel GH$, 且 $EF = GH$,

$\therefore EFHG$ 平行四边形, $\therefore FH \parallel EG$, $\dots\dots\dots 1 \text{ 分}$

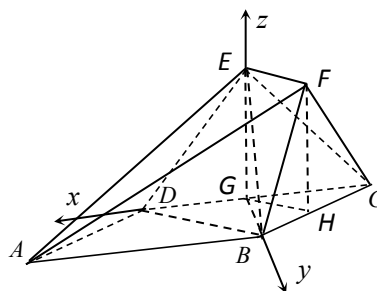
$\therefore CE = DE$, $\therefore EG \perp CD$,

$\therefore$ 平面 $CDE \perp$ 平面 $ABCD$,

$\therefore EG \perp$ 平面 $ABCD$, $\dots\dots\dots 3 \text{ 分}$

$\therefore FH \perp$ 平面 $ABCD$, $\therefore FH \subset$ 平面 BCF ,

$\therefore$ 平面 $BCF \perp$ 平面 $ABCD$; $\dots\dots\dots 6 \text{ 分}$



(II) 连结 BG , 由 (I) 得 $EG \perp$ 平面 $ABCD$,

$\therefore$ 四边形 $ABCD$ 是边长为 2 的菱形, 且 $\angle BAD = 60^\circ$,

$\therefore \triangle ABC$ 是边长为 2 的等边三角形, $\therefore BG \perp CD$, $BG = \sqrt{3}$, $\dots\dots\dots 8 \text{ 分}$

以 G 为坐标原点, 向量 $\overrightarrow{GD}, \overrightarrow{GB}$ 的方向分别为 x 轴, y 轴的正方向, 建立如图所示的空间直角坐标系 $G-xyz$, 由题意得 $G(0,0,0)$, $A(2, \sqrt{3}, 0)$, $B(0, \sqrt{3}, 0)$, $C(-1, 0, 0)$,

设 $E(0, 0, t) (t > 0)$, 则 $F(-\frac{1}{2}, \frac{\sqrt{3}}{2}, t)$,

设 $\vec{m} = (x_1, y_1, z_1)$ 是平面 AEF 的一个法向量,

$$\text{则 } \begin{cases} \vec{m} \cdot \overrightarrow{EF} = 0, \\ \vec{m} \cdot \overrightarrow{AE} = 0, \end{cases} \therefore \begin{cases} \frac{1}{2}x_1 - \frac{\sqrt{3}}{2}y_1 = 0, \\ 2x_1 + \sqrt{3}y_1 - tz_1 = 0, \end{cases} \text{ 令 } y_1 = 1, \text{ 则 } \begin{cases} x_1 = \sqrt{3}, \\ z_1 = \frac{3\sqrt{3}}{t}, \end{cases} \therefore \vec{m} = (\sqrt{3}, 1, \frac{3\sqrt{3}}{t}),$$

设 $\vec{n} = (x_2, y_2, z_2)$ 是平面 BCF 的一个法向量,

$$\text{则 } \begin{cases} \vec{m} \cdot \overrightarrow{BC} = 0, \\ \vec{m} \cdot \overrightarrow{BF} = 0, \end{cases} \therefore \begin{cases} x_2 + \sqrt{3}y_2 = 0, \\ \frac{1}{2}x_2 + \frac{\sqrt{3}}{2}y_2 - tz_2 = 0, \end{cases} \text{ 令 } y_2 = -1, \text{ 则 } \begin{cases} x_2 = \sqrt{3}, \\ z_2 = 0, \end{cases} \therefore \vec{n} = (\sqrt{3}, -1, 0),$$

$$\therefore |\cos \langle \vec{m}, \vec{n} \rangle| = \frac{|\vec{m} \cdot \vec{n}|}{|\vec{m}| |\vec{n}|} = \frac{1}{\sqrt{4 + \frac{27}{t^2}}} = \frac{\sqrt{5}}{5}, \therefore t = 3\sqrt{3}, \quad \dots\dots\dots 11 \text{ 分}$$

$$\therefore GE = 3\sqrt{3}, \therefore \text{直线 } BE \text{ 与平面 } ABCD \text{ 所成角的正弦值 } \frac{3\sqrt{10}}{10}. \quad \dots\dots\dots 12 \text{ 分}$$

$$20 \text{ 解: (I) 设 } D(\frac{2}{3}, y_0), \text{ 由题意得 } \begin{cases} k_{DA} \cdot k_{DB} = \frac{y_0}{\frac{2}{3}+a} \cdot \frac{y_0}{\frac{2}{3}-a} = -\frac{1}{4}, \\ \frac{1}{2} \times 2a \times |y_0| = \frac{4\sqrt{2}}{3}, \\ \frac{4}{9a^2} + \frac{y_0^2}{b^2} = 1, \end{cases} \dots\dots\dots 3 \text{ 分}$$

$$\therefore \begin{cases} b^2 = 1, \\ a^2 = 4, \end{cases} \therefore \text{椭圆 } C \text{ 的方程为 } \frac{x^2}{4} + y^2 = 1; \dots\dots\dots 5 \text{ 分}$$

(II) 假设存在这样的点 N , 设直线 PM 与 x 轴相交于点 $T(x_0, 0)$, 由题意得 $TP \perp BQ$,
 由 (I) 得 $B(2, 0)$, 设 $P(\frac{2}{3}, t)$, $Q(x_1, y_1)$, 由题意可设直线 AP 的方程为 $x = my - 2$,

$$\text{由 } \begin{cases} x = my - 2, \\ \frac{x^2}{4} + y^2 = 1 \end{cases} \text{ 得 } (m^2 + 4)y^2 - 4my = 0, \therefore y_1 = \frac{4m}{m^2 + 4} \text{ 或 } y_1 = 0 \text{ (舍去)}, x_1 = \frac{2m^2 - 8}{m^2 + 4}, \dots\dots\dots 7 \text{ 分}$$

$$\therefore \frac{2}{3} = mt - 2, \therefore t = \frac{8}{3m},$$

$$\therefore TP \perp BQ, \therefore \overrightarrow{TP} \cdot \overrightarrow{BQ} = (\frac{2}{3} - x_0)(x_1 - 2) + ty_1 = 0,$$

$$\therefore x_0 = \frac{2}{3} + \frac{ty_1}{x_1 - 2} = \frac{2}{3} + \frac{8}{3m} \cdot \frac{4m}{m^2 + 4} \cdot \frac{m^2 + 4}{-16} = 0, \dots\dots\dots 10 \text{ 分}$$

$\therefore$ 直线 PM 过定点 $T(0, 0)$,

$\therefore$ 存在定点 $N(1, 0)$, 使得 $|MN| = 1$. $\dots\dots\dots 12 \text{ 分}$

$$21. \text{ (I) 解: 令 } h(x) = f(x) - g(x) = \sqrt{x^2 + 1} + x - \sin x - \cos x, \quad x \geq -\frac{\pi}{4},$$

$$(1) \text{ 当 } -\frac{\pi}{4} \leq x < \frac{\pi}{4} \text{ 时, 则 } h'(x) = \frac{x}{\sqrt{x^2 + 1}} + 1 - \cos x + \sin x,$$

$$\therefore h''(x) = \frac{1}{(\sqrt{x^2 + 1})^{\frac{3}{2}}} + \sqrt{2} \sin(x + \frac{\pi}{4}) > 0, \dots\dots\dots 1 \text{ 分}$$

$$\therefore h'(x) \text{ 在 } [-\frac{\pi}{4}, \frac{\pi}{4}) \text{ 上单调递增, 且 } h'(0) = 0,$$

$$\text{当 } -\frac{\pi}{4} \leq x < 0 \text{ 时, } h'(x) < 0; \text{ 当 } 0 \leq x < \frac{\pi}{4} \text{ 时, } h'(x) \geq 0,$$

$$\therefore h(x) \text{ 在 } [-\frac{\pi}{4}, 0) \text{ 上递减, 在 } [0, \frac{\pi}{4}) \text{ 上递增, } \therefore h(x) \geq h(0) = 0, \therefore f(x) \geq g(x); \dots\dots\dots 3 \text{ 分}$$

$$(2) \text{ 当 } x \geq \frac{\pi}{4} \text{ 时, 则 } h(x) = \sqrt{x^2+1} + x - \sqrt{2} \sin(x + \frac{\pi}{4}) \geq \sqrt{x^2+1} + x - \sqrt{2} \\ \geq \sqrt{(\frac{\pi}{4})^2+1} + \frac{\pi}{4} - \sqrt{2} > 1 + \frac{\pi}{4} - \sqrt{2} > 0, \therefore f(x) \geq g(x);$$

综上所述, 当 $x \geq -\frac{\pi}{4}$ 时, $f(x) \geq g(x)$;5 分

$$(II) \text{ 令 } t(x) = f(x) + g(x) - ax - 2 = \sqrt{x^2+1} + x + \sin x + \cos x - ax - 2, \quad x \geq 0,$$

$$\text{则 } t'(x) = \frac{x}{\sqrt{x^2+1}} + 1 + \cos x - \sin x - a,$$

由题意得 $t(x) \leq 0$ 在 $[0, +\infty)$ 上恒成立, $\therefore t(0) = 0, \therefore t'(0) = 2 - a \leq 0, \therefore a \geq 2$;7 分
下证当 $a \geq 2$ 时, $t(x) \leq 0$ 在 $[0, +\infty)$ 上成立,

$$\therefore t(x) = \sqrt{x^2+1} + x + \sin x + \cos x - ax - 2 \leq \sqrt{x^2+1} + x + \sin x + \cos x - 2x - 2,$$

令 $\varphi(x) = \sqrt{x^2+1} - x + \sin x + \cos x - 2$, 只需证明 $\varphi(x) \leq 0$ 在 $[0, +\infty)$ 上成立,8 分

$$(1) \text{ 当 } 0 \leq x \leq \frac{\pi}{4} \text{ 时, } \varphi'(x) = \frac{x}{\sqrt{x^2+1}} - 1 + \cos x - \sin x,$$

$$\varphi''(x) = \frac{1}{(\sqrt{x^2+1})^{\frac{3}{2}}} - \sqrt{2} \sin(x + \frac{\pi}{4}), \therefore \varphi''(x) \text{ 在 } [0, \frac{\pi}{4}] \text{ 上单调递减, } \therefore \varphi''(x) \leq \varphi''(0) = 0,$$

$$\therefore \varphi'(x) \text{ 在 } [0, \frac{\pi}{4}] \text{ 上单调递减, } \therefore \varphi'(x) \leq \varphi'(0) = 0,$$

$$\therefore \varphi(x) \text{ 在 } [0, \frac{\pi}{4}] \text{ 上单调递减, } \therefore \varphi(x) \leq \varphi(0) = 0; \quad \text{.....10 分}$$

$$(2) \text{ 当 } x > \frac{\pi}{4} \text{ 时, } \varphi(x) = \sqrt{x^2+1} - x + \sqrt{2} \sin(x + \frac{\pi}{4}) - 2 \leq \sqrt{x^2+1} - x + \sqrt{2} - 2$$

$$\leq \sqrt{(\frac{\pi}{4})^2+1} - \frac{\pi}{4} + \sqrt{2} - 2 < 0;$$

综上所述, 实数 a 的取值范围是 $[2, +\infty)$12 分

$$22 \text{ 解: (I) 将 } \begin{cases} x = \frac{2\sqrt{2}t}{t^2+1}, \\ y = \frac{t^2-1}{t^2+1} \end{cases} \text{ 的参数 } t \text{ 消去得曲线 } C \text{ 的普通方程为 } \frac{x^2}{2} + y^2 = 1 (y \neq 1), \quad \text{.....3 分}$$

$$\therefore \rho \cos(\theta + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \therefore \rho \cos \theta - \rho \sin \theta - 1 = 0,$$

$$\text{由 } \begin{cases} x = \rho \cos \theta, \\ y = \rho \sin \theta \end{cases} \text{ 可得直线 } l \text{ 的直角坐标方程为 } x - y - 1 = 0; \quad \text{.....5 分}$$

(II) 由(I)得曲线C的参数方程可表示为 $\begin{cases} x = \sqrt{2} \cos \theta, \\ y = \sin \theta \end{cases}$ (θ 为参数) ($\theta \neq \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$),

设 $A(\sqrt{2} \cos \theta, \sin \theta)$, 则点A到直线l的距离 $d = \frac{|\sqrt{2} \cos \theta - \sin \theta - 1|}{\sqrt{2}} = \frac{\sqrt{2}}{2}$,7分

$\therefore \sqrt{2} \cos \theta - \sin \theta = 0$ 或 $\sqrt{2} \cos \theta - \sin \theta = \sqrt{3} \cos(\theta + \varphi) = 2$ ($\tan \varphi = \sqrt{2}$) (舍去),

$\therefore \sin^2 \theta + \cos^2 \theta = 1$, $\therefore \cos \theta = \pm \frac{\sqrt{3}}{3}$,

当 $\cos \theta = \frac{\sqrt{3}}{3}$ 时, $A(\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{3})$; 当 $\cos \theta = -\frac{\sqrt{3}}{3}$ 时, $A(-\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{3})$10分

23 解: (I) 当 $m = 1$ 时, 原不等式为 $|x+1| + |2x-1| \leq 6$,

$\begin{cases} x < -1, \\ -(x+1) - (2x-1) \leq 6 \end{cases}$ 或 $\begin{cases} -1 \leq x \leq \frac{1}{2}, \\ x+1 - (2x-1) \leq 6 \end{cases}$ 或 $\begin{cases} x > \frac{1}{2}, \\ x+1 + 2x-1 \leq 6, \end{cases}$ 3分

$\therefore -2 \leq x < -1$ 或 $-1 \leq x \leq \frac{1}{2}$ 或 $\frac{1}{2} < x \leq 2$

$\therefore$ 原不等式 $f(x) \leq 6$ 的解集为 $\{x | -2 \leq x \leq 2\}$;5分

(II) 由题意得 $f(x) = \begin{cases} -3x - m^2 + m, & x < -m^2, \\ -x + m^2 + m, & -m^2 \leq x \leq \frac{m}{2}, \\ 3x + m^2 - m, & x > \frac{m}{2}, \end{cases}$

$\therefore f(x)_{\min} = f(\frac{m}{2}) = m^2 + \frac{1}{2}m = \frac{3}{2}$, $\therefore m = 1$ 或 $m = -\frac{3}{2}$ (舍去),7分

$\therefore a + b = 1$, 令 $\begin{cases} a = \cos^2 \theta, \\ b = \sin^2 \theta \end{cases}$ ($0 < \theta < \frac{\pi}{2}$),

则 $\sqrt{a} + 2\sqrt{b} = \cos \theta + 2\sin \theta = \sqrt{5} \sin(\theta + \varphi) \leq \sqrt{5}$,

当 $\theta = \frac{\pi}{2} - \varphi$ ($0 < \varphi < \frac{\pi}{2}$, 且 $\tan \varphi = \frac{1}{2}$) 时, 上述不等式取等号.10分

注: 以上各题其他解法, 请酌情给分.